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Quantum δ-kicked rotor: the effect of amplitude noise on the quantum resonances

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Abstract

We study analytically the effect of amplitude noise on the quantum resonances of an atom optics realization of the δ-kicked rotor. Noise is shown to add a time growth to the 'deterministic' energy and to induce a time increasing spreading in the momentum distribution; exact results are given for both effects. The ballistic peaks characteristic of the noiseless distribution for particular initial conditions broaden and eventually vanish, whereas the associated quadratic growth of energy persists; at large times, the survival probability decays as t^{-1}. Moreover, the nonexponential 'localization' linked to different initial conditions is gradually destroyed. Features specific to Gaussian noise, white and coloured, are analysed. The feasibility of experimental tests of these effects is discussed.

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1. Introduction

Recent research in atom optics has allowed experimental study of a variety of fundamental effects. Especially interesting is the realization of the δ-kicked rotor [1–3], a paradigm in classical and quantum chaos [4–6], achieved with a two-level atom interacting with a pulsed standing light wave. Predictions on aspects of this system relevant to issues such as quantum–classical correspondence or anomalous diffusion have been tested. In particular, *dynamical localization*, i.e. the quantum suppression of chaos reflected in an exponential localization of the probability distribution, has been verified [2, 3]. Moreover, decoherence effects [7–10] have been observed: noise has been shown to induce the destruction of localization and the partial recovery of the classical behaviour [11–13].

In the present paper, we focus on another important aspect of the dynamics of the δ-kicked rotor, namely, the *quantum resonances* (QR) [5, 14, 15]. These are intrinsically quantum features which result from well-chosen values of the kicking period and are characterized by *ballistic motion* (BM), i.e. quadratic time increase in the energy, for particular initial conditions. The role of the initial preparation of the system in the occurrence of this effect has been clarified by recent experiments; remarkably, the momentum distribution has been shown to present nonexponential 'localization' instead of ballistic growth for quite general initial conditions [3, 14]. We aim at discussing the robustness of these features against fluctuations in the *stochasticity parameter* of the rotor. Apart from its intrinsic fundamental interest, the problem has practical implications: these fluctuations, which will be termed *amplitude noise* [11, 12], correspond to systematic noise sources in the realization of the model; in particular, to random drifts in the intensity of the standing wave. Furthermore, since different types of noise can be incorporated into the experimental set-up, the system provides a scenario for analysing the effect of a controllable decohering mechanism on specifically quantum features. Indeed, the interest of studying the relevance of the noise statistical characteristics to the coherent evolution of the system has been emphasized in recent work [11, 12]. In contrast with the complexity of the studies of dynamical localization, we present an exact analytical treatment, feasible at QR due to the simplicity of the evolution between kicks. Our exact results for the energy and the momentum distribution give insight into nontrivial aspects of the decoherence phenomenology such as nonexponential decay [16], persistence of quantum characteristics in the open system, or effects specific to different random drivings.

We focus on the realization of the δ-kicked rotor reported in [2, 3]. In the absence of noise, the Hamiltonian, in the dimensionless notation of [3], reads

$$H(\phi, \rho, \tau) = \frac{\rho^2}{2} + K\cos\phi \sum_{n=-\infty}^{\infty} \delta(\tau - n) \tag{1}$$

where the conjugate variables ϕ and ρ obey $[\phi, \rho] = \mathrm{i}\bar{k}$, $\bar{k}$ being a scaled Planck constant. It is assumed that the pulses can be approximated as δ-kicks with an effective stochasticity parameter K. In this description, K and $\bar{k}$ completely determine the quantum dynamics. We concentrate on the QR defined by $\bar{k}/2 = q2\pi$, where q is an integer; as the classical dynamics is solely determined by K, the features resulting from this condition have specifically quantum nature. In this regime, an initial state $\psi_0(\phi) = \exp[\mathrm{i}(n_0 + \nu_0)\phi]$, where n_0 is an integer and $\nu_0 \in [-1/2, 1/2)$, evolves after N kicks into [10]

$$\psi_N(\phi) = \exp\left[\mathrm{i}(n_0 + \nu_0)(\phi - N\bar{k}\nu_0) - \mathrm{i}\frac{K}{\bar{k}} \sum_{l=0}^{N-1} \cos[\phi - (N-l)\bar{k}\nu_0]\right]. \tag{2}$$

Note that states with $\nu_0 \neq 0$ are considered in our study since they are relevant to the atomic realization of the δ-kicked rotor. The mean energy is obtained as $\langle E(N)\rangle \equiv \langle \rho^2/2\rangle = \frac{\bar{k}^2}{2}(n_0 + \nu_0)^2 + \frac{\bar{K}^2}{4}\left(\frac{\sin N\beta_0}{\sin\beta_0}\right)^2$, with $\beta_0 \equiv \bar{k}\nu_0/2$; additionally, the probability of having a momentum $\rho = (n + \nu_0)\bar{k}$ is $J^2_{n-n_0}\left(\frac{\bar{K}}{\bar{k}}\frac{\sin N\beta_0}{\sin\beta_0}\right)$, where $J_m(x)$ are the Bessel functions [17]. For $\nu_0 = 0, -1/2$, the energy increases quadratically with N. Moreover, since $J^2_{\pm n}(x)$ peaks when x is close to n, two ballistic peaks appear at the edges of the momentum distribution [14]; the sample is, therefore, accelerated in a highly nonuniform way. For $\nu_0 \neq 0, -1/2$, the energy 'oscillates'; the distribution shows partial or complete revivals. BM is still observed at small times in the limit $\nu_0 \to 0, -1/2$; outside this limit, quasilocalization sets in [3]. As opposed to the exponential character of dynamical localization, the 'static' distribution at QR has a nonexponential profile. Our objective is to investigate how this behaviour is affected by amplitude noise. First, a plane-wave initial state, $\psi_0(\phi) = \exp[\mathrm{i}(n_0 + \nu_0)\phi]$, will be assumed; despite its simplicity, its evolution shows the key features of the problem. Next, the generalization to realistic states which correspond to experimental realizations of the system will be presented.

2. The effect of noise on energy

Amplitude noise is introduced through $K = \bar{K} + \delta K$, where $\bar{K}$ is the mean value of the stochasticity parameter and δK corresponds to a zero-mean random variable. Following standard methodology [18, 19], we first obtain the state after N kicks for each stochastic realization, which reads [3]

$$\psi_N(\phi) = \exp\left[\mathrm{i}(n_0 + \nu_0)(\phi - N\bar{k}\nu_0) - \frac{\mathrm{i}}{\bar{k}}\sum_{l=0}^{N-1}(\bar{K} + \delta K_l)\cos[\phi - (N-l)\bar{k}\nu_0]\right] \tag{3}$$

where δK_l denotes the value taken by δK at the lth kick. The mean energy $\langle E(N)\rangle$ is readily calculated; the subsequent average over fluctuations ($\langle\cdots\rangle_f$) yields

$$\langle\langle E(N)\rangle\rangle_f = \frac{\bar{k}^2(n_0 + \nu_0)^2}{2} + \left(\frac{\bar{K}}{2}\frac{\sin N\beta_0}{\sin\beta_0}\right)^2 + E^{(f)} \tag{4}$$

where, added to the initial value $\bar{k}^2(n_0 + \nu_0)^2/2$, two differently rooted contributions are apparent. The second term, which gives the energy enhancement corresponding to a noiseless system with stochasticity parameter $\bar{K}$, implies the persistence of some deterministic features in the random dynamics, in particular, of the BM linked to states with $\nu_0 = 0, -1/2$. We stress that BM is rooted in the nondispersive evolution between kicks allowed by the QR condition; importantly, this nondispersive character is preserved by amplitude noise. The third term, defined as

$$E^{(f)} \equiv \left\langle\left\langle \frac{1}{2}\left[\sum_{l=0}^{N-1}\delta K_l \sin[\phi - (N-l)\bar{k}\nu_0\right]^2\right\rangle\right\rangle_f \tag{5}$$

reflects a noise-induced growth of the energy which modifies the deterministic dependence; the corresponding change in the momentum distribution will be shown to gradually broaden the ballistic peaks observed for $\nu_0 = 0, -1/2$ and the nonexponentially 'localized' distribution detected outside the limit $\nu_0 \to 0, -1/2$. No restrictions on the statistics of δK have been introduced so far. Let us now analyse features specific to different stochastic characteristics. For Gaussian white noise, namely, for noise with $\langle\delta K(\tau)\rangle_f = 0, \langle\delta K(\tau)\delta K(\tau')\rangle_f = \mathrm{Var}(\delta K)\delta(\tau - \tau')$ and higher order moments given by the standard factorization form (see, for instance, [20]), we find

$$E^{(f)} = \frac{1}{2\pi}\int \mathrm{d}\phi \sum_{l=0}^{N-1}\left\langle\delta K_l^2\right\rangle_f \sin^2[\phi - (N-l)\bar{k}\nu_0] = \frac{1}{4}\mathrm{Var}(\delta K)N \tag{6}$$

which shows that a purely diffusive spread is added to the deterministic energy. As only up to two-time correlations intervene in the averaging, a linear dependence on N and $\mathrm{Var}(\delta K)$ can also be predicted for non-Gaussian white noise. Of interest for the general discussion on the role of fluctuations in the quantum–classical correspondence (see [3] and references therein) is that, in the QR regime considered in our system, amplitude noise does not induce the emergence of classical behaviour: the deterministic contribution, which has a specifically quantum character, and the noisy term are simultaneously present at any time and irrespective of the noise intensity. At this point, it is worth recalling that the limit $\bar{k} \to 0$ is incompatible with the QR condition.

Let us now analyse the effect of coloured noise. For exponentially correlated fluctuations, i.e. for $\langle\delta K(\tau)\delta K(\tau')\rangle = \mathrm{Var}(\delta K)\exp(-|\tau - \tau'|/\tau_c)$ [20], our previous derivation is

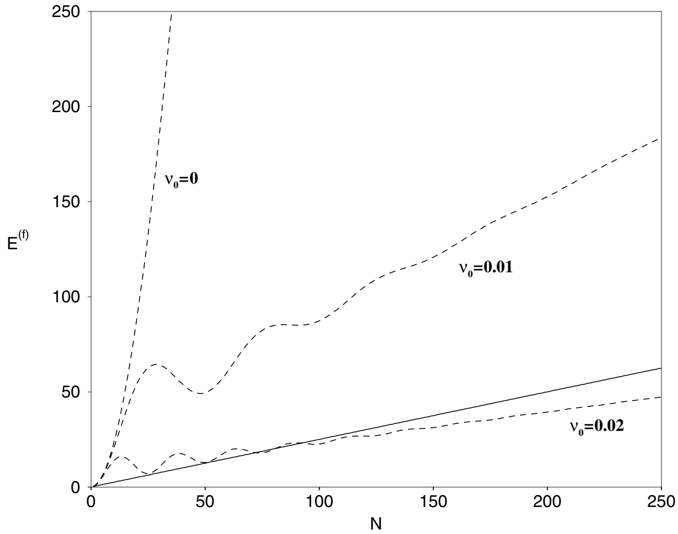

Figure 1. The contribution of noise to the mean energy, $E^{(f)}$, versus N, for different initial states. The dashed lines correspond to coloured noise with $\tau_c = 50$; the solid line corresponds to white noise (equation (6)). $\mathrm{Var}(\delta K) = 1$.

generalized to obtain

$$E^{(f)} = \frac{\mathrm{Var}(\delta K)}{4} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \exp(-|m-l|/\tau_c) \cos(l-m) 2\beta_0 = \frac{\mathrm{Var}(\delta K)}{4} \frac{\sinh \gamma_c}{\cosh \gamma_c - \cos 2\beta_0}$$
$$\times \left[N + \frac{(\cosh \gamma_c \cos 2\beta_0 - 1)(\mathrm{e}^{-N/\tau_c} \cos 2N\beta_0 - 1) - \mathrm{e}^{-N/\tau_c} \sin 2N\beta_0 \sin 2\beta_0 \sinh \gamma_c}{\sinh \gamma_c (\cosh \gamma_c - \cos 2\beta_0)} \right] \quad (7)$$

where $\gamma_c \equiv 1/\tau_c$. Two significant differences with the white noise case are evident: the nonlinearity of the time increase and the dependence on the initial state (see figure 1); indeed, maxima of $E^{(f)}$ are reached at $\nu_0 = 0, -1/2$. Furthermore, as illustrated in figure 2, $E^{(f)}$ presents a nontrivial dependence on the correlation time. The origin of these features is clarified by the analysis of the following limiting cases:

(i) For $N \ll \tau_c$, which corresponds to the initial behaviour for long correlation times, one finds $E^{(f)} = \frac{\mathrm{Var}(\delta K)}{4} \left(\frac{\sin N\beta_0}{\sin \beta_0} \right)^2$, which has the same functional form as the deterministic contribution. In this regime, as the noise correlations have hardly decayed, the system behaves in a purely deterministic way with total stochasticity parameter given by $K = [\bar{K}^2 + \mathrm{Var}(\delta K)]^{1/2}$.

(ii) The asymptotic behaviour is trivially characterized: for $N \gg 1, \tau_c$, a linear spreading emerges as the noise-induced term reads

$$E^{(f)} = \frac{1}{4} \mathrm{Var}(\delta K) \frac{\sinh \gamma_c}{\cosh \gamma_c - \cos 2\beta_0} N. \quad (8)$$

Hence, in this limit, the effect of coloured noise parallels that of an effective white noise. A simplified description of this behaviour is given by a coarse-grained picture of the dynamics with time scale defined by the number M of correlated kicks ($N \gg M$). Since the correlations are lost after M kicks, coloured noise is described in this approach as an effective white noise with intensity depending on the dynamics inside the coarse-graining period.

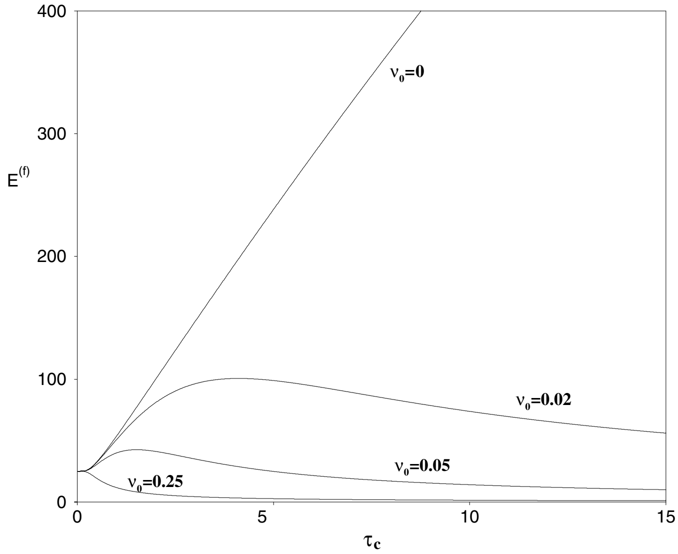

Figure 2. $E^{(f)}$ versus τ_c for different initial states. $N = 100$ and $\text{Var}(\delta K) = 1$.

Important features apparent in the above expression give additional insight into this regime. Traces of deterministic behaviour can still be observed in the dependence of the effective white-noise variance on β_0. Indeed, qualitative changes in the dependence of $E^{(f)}$ on noise colour are observed as the initial condition is varied; e.g., for $\nu_0 = 0, -1/2$, we find $E^{(f)} = \frac{1}{4}\,\text{Var}(\delta K)\coth(\gamma_c/2)N$, which reflects a colour-induced enhancement of the response; significantly, these initial conditions correspond to ballistic growth in the deterministic system. In contrast, for $\nu_0 = 1/4$, which corresponds to deterministic localization, we obtain $E^{(f)} = \frac{1}{4}\,\text{Var}(\delta K)\tanh(\gamma_c)N$, where the opposite behaviour is apparent. As expected, the traces of the initial condition in $E^{(f)}$ vanish as the white-noise limit is approached; for $\tau_c \to 0$, we consistently recover $E^{(f)} = \frac{1}{4}\,\text{Var}(\delta K)N$. This distinctive effect of coloured noise can be understood from an analysis of equation (7): the dependence of $E^{(f)}$ on β_0, i.e. on the initial condition, becomes apparent as soon as there is a correlation between the values taken by δK at two consecutive kicks.

From these results, it follows that coloured-noise effects in our system can be intuitively depicted as intermediate between the purely deterministic behaviour observed in the initial transient and the white-noise effects emerging in the asymptotic regime. Actually, traces of characteristics of the two limits can be identified in features specific to the finite bandwidth of the fluctuations.

3. The momentum distribution

Let us complete our picture of the dynamics with the analysis of the momentum distribution. By expanding equation (3) in products of Bessel functions and applying the 'summation theorems' [21], the coefficients of the wavefunction in the momentum representation, $a_n^{(N)}$, are obtained as functions with the same form as their deterministic counterparts and stochastic arguments: $a_n^{(N)} \propto J_{n-n_0}(\eta_N)$, where, for generic noise, η_N is defined through

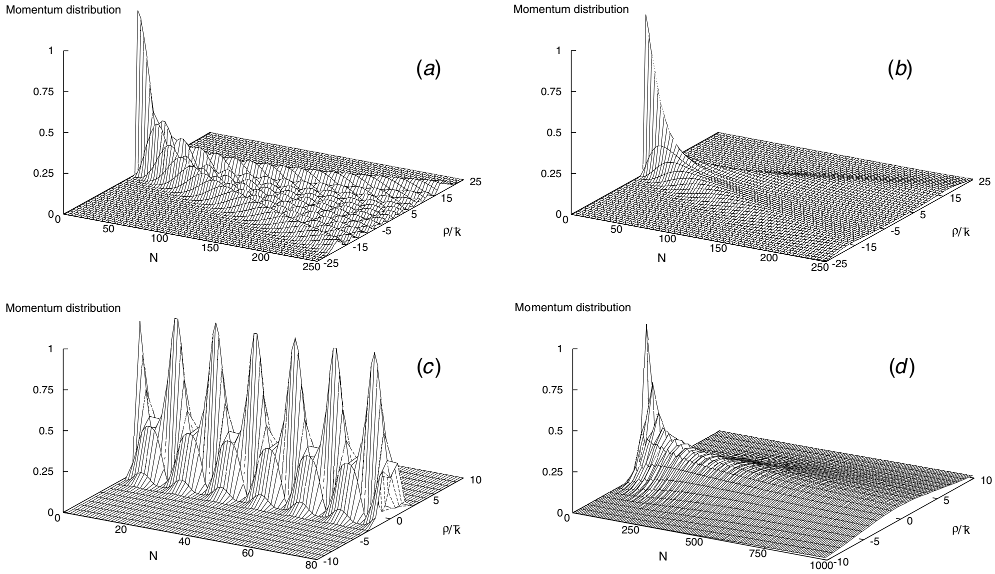

Figure 3. Evolution of the momentum distribution for (*a*) $\nu_0 = 0$ and $K = 1$, in the noiseless case; (*b*) $\nu_0 = 0$ and $\bar{K} = 1$, for white noise; (*c*) $\nu_0 = 0.04$ and $K = 5$, in the noiseless case; and (*d*) $\nu_0 = 0.04$ and $\bar{K} = 5$, for white noise. $\mathrm{Var}(\delta K) = 9$.

$$\eta_N = \left(z_N^2 + \sum_{l=0}^{N-1} \sum_{j=0}^{N-1} \frac{\delta K_l}{\bar{k}} \frac{\delta K_j}{\bar{k}} \cos(l-j)2\beta_0 + 2 z_N \sum_{l=0}^{N-1} \frac{\delta K_l}{\bar{k}} \cos(N-1-2l)\beta_0 \right)^{1/2} \tag{9}$$

with $z_N = \frac{\bar{K}}{\bar{k}} \frac{\sin N\beta_0}{\sin \beta_0}$. Hence, the probability of having a momentum $\rho = (n + \nu_0)\bar{k}$ is $\left\langle \left| a_n^{(N)} \right|^2 \right\rangle_f = \left\langle J_{n-n_0}^2(\eta_N) \right\rangle_f$. Note that, as opposed to the mean energy, the momentum distribution can be affected by noise correlations of order higher than two. Equation (9) reflects that the number of noisy terms which are added to give η_N increases with N. Therefore, we can predict an increasing dispersion of η_N and, as a result, a noise induced spreading of the distribution growing with N and $\mathrm{Var}(\delta K)$; indeed, the attenuation of the deterministic features due to the averaging in $\left\langle J_{n-n_0}^2(\eta_N) \right\rangle_f$, in particular, the gradual destruction of the localization, can be anticipated. Let us put this discussion on a more quantitative ground.

(a) For $\nu_0 = 0$, one has $\eta_N = z_N + \sum_{l=0}^{N-1} \frac{\delta K_l}{\bar{k}}$, which, for Gaussian white noise, is characterized by $\langle \eta_N(\tau) \rangle = z_N$ and $\langle \eta_N(\tau)\eta_N(\tau') \rangle = N\,\mathrm{Var}(\delta K)\bar{k}^{-2}\delta(\tau - \tau')$; these properties correspond also to η_N for $\nu_0 = -1/2$. In these cases, the averages are given by

$$\left\langle \left| a_n^{(N)} \right|^2 \right\rangle_f = \frac{1}{\sqrt{\frac{\pi}{2}\,\mathrm{Var}(\delta K) N}} \int_{-\infty}^{\infty} J_{n-n_0}^2(\eta_N) \exp\left[-\frac{(\eta_N - z_N)^2}{\frac{1}{2}\,\mathrm{Var}(\delta K) N} \right] \mathrm{d}\eta_N. \tag{10}$$

The results for $\nu_0 = 0$, for the deterministic system and for its stochastic parallel, are presented in figures 3(*a*) and (*b*), respectively. There it is shown how noise broadens the ballistic peaks; although BM persists at any time (see equation (4)), the peak structure in the distribution is hardly noticeable at large N. An understanding of these findings is given by the following arguments. First, one should recall that the ballistic peaks are rooted in the maxima reached by the functions $J_{\pm n}^2(x)$ at x close to n. In the deterministic system, for $\nu_0 = 0$, the argument z_N increases linearly with N and, therefore, the two maxima move to higher (negative and positive) values of the momentum; moreover, because of the properties of the Bessel functions,

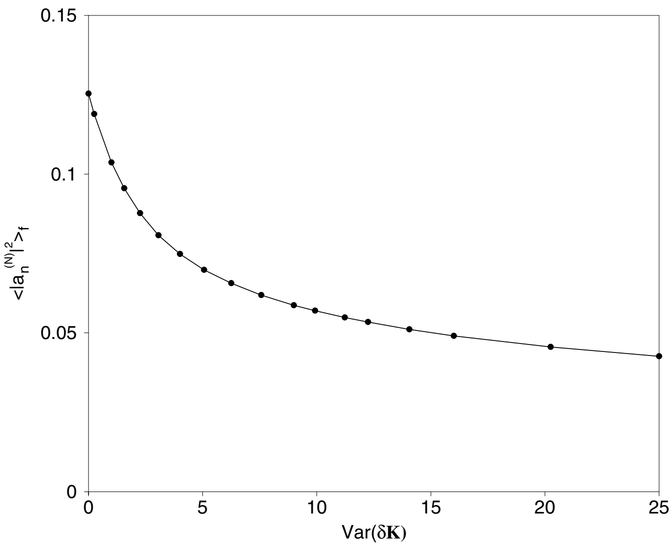

Figure 4. The coefficient of the momentum distribution at a ballistic peak $\langle |a_n^{(N)}|^2\rangle_f$ ($N = 94$, $n = 6$) versus the variance, in the case of white noise.

an increasing number of intermediate peaks appear as z_N grows. Second, in the noisy system, the argument η_N is a stochastic variable with mean value z_N and a variance which is proportional to both the noise intensity and the number of kicks. The implications for the ballistic peaks are clear: the decreasing probability $1\Big/\sqrt{\frac{\pi}{2}\,\mathrm{Var}(\delta K)N}$ of having $\eta_N = z_N$ implies a smaller contribution of the maximum $J_n^2(z_N)$ to the average in equation (10); moreover, as values of η_N larger and smaller than z_N become increasingly probable, the statistical weights of smaller values of $J_n^2(\eta_N)$ increase. As a consequence, the coefficients $\left\langle \left|a_n^{(N)}\right|^2\right\rangle_f$ associated with the ballistic peaks decrease. Additionally, the dispersion of η_N allows that the higher order Bessel functions that correspond to coefficients beyond the peak, which take almost zero values at z_N, now reach values sufficiently important to make larger momenta become apparent in the distribution.

Further insight is obtained by focusing separately on two important points of the above discussion: the dependence of the spreading on the noise intensity and the time evolution of the coefficients. Figure 4 depicts, at fixed N, the noise-induced decay of the coefficient $\left\langle \left|a_n^{(N)}\right|^2\right\rangle_f$ corresponding to a ballistic peak. The dependence of the peak height on the noise variance apparent in that figure, which corresponds to a decrease slower than $[\mathrm{Var}(\delta K)]^{-1/2}$, is qualitatively explained by an analysis of equation (10): one can easily check that, because of the properties of $J_n^2(x)$, the integral grows with the variance much more slowly than the prefactor decay. On the other hand, the dependence of the coefficients on N, rooted not only in the stochastic variation of η_N, but also in the deterministic drift of z_N, presents a more complex appearance. An analytical description of this dependence can be given at large times. In this limit, taking into account the asymptotic behaviour of the Bessel functions, one obtains $J_n^2(z_N) \sim N^{-1}\cos^2(z_N - n\pi/2 - \pi/4)$ for the deterministic system and $\left\langle J_n^2(\eta_N)\right\rangle_f \sim N^{-1}$ for its stochastic counterpart. The persistence in the random dynamics of part of the deterministic functional form can be understood: the mean value of η_N grows with N faster than its stochastic dispersion (z_N and the variance both increase linearly); therefore, at large N, $J_n^2(\eta_N)$ is always inside its asymptotic regime [17]. The oscillations, which are the

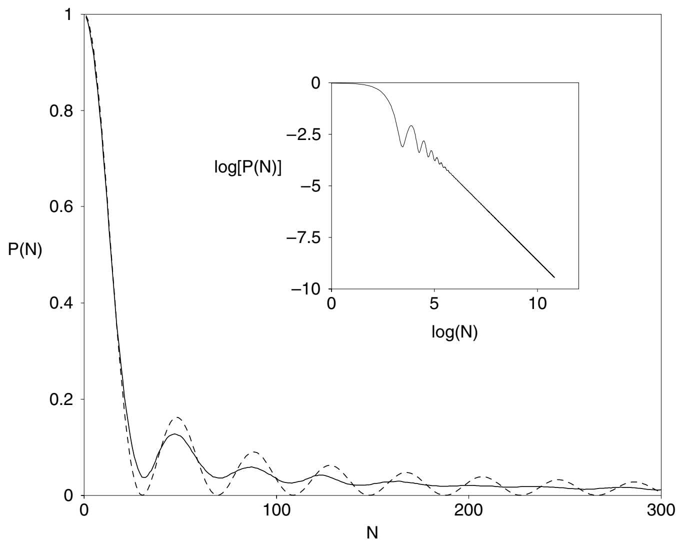

Figure 5. The survival probability, $P(N)$, versus the number of kicks for white noise with $\mathrm{Var}(\delta K) = 1$ (solid line). The dashed line corresponds to the noiseless system. The inset shows a logarithmic plot of the noisy asymptotic regime. $\bar{K} = 1$ and $\nu_0 = 0$.

signature of the residual peak structure, are washed out by the averaging in $\left\langle J_n^2(\eta_N)\right\rangle_f$; the dependence on N^{-1} remains.

The above argument explains, in particular, the behaviour of the 'survival probability', $P(N) \equiv \langle|\langle\psi_0(\phi)\psi_N(\phi)\rangle|^2\rangle_f = \left\langle J_0^2(\eta_N)\right\rangle_f$, which decays as N^{-1} at large times, as illustrated in figure 5. There it is shown that, after an initial transient in which there is a small difference between the random and the noiseless functions, the oscillations decay and vanish much faster than the final N^{-1} decrease. In order to isolate the effects specific to noise, we have taken $\bar{K} = 0$ (as K can be negative, this case corresponds in fact to a mixture of amplitude and phase noises). Since $z_N = 0$, a purely diffusive spreading takes place; the survival probability being given by

$$\left\langle J_0^2(\eta_N)\right\rangle_f = \sum_{p=0}^{\infty}\left(\frac{(2p-1)!!}{p!}\right)^2 \frac{(-1)^p}{p!}\left[\frac{\mathrm{Var}(\delta K)N}{2\bar{k}^2}\right]^p \tag{11}$$

which corresponds to exponential decay only for small values of $\mathrm{Var}(\delta K)N/2\bar{k}^2$. Note that, in this case, there is a common dependence on $\mathrm{Var}(\delta K)$ and N, as opposed to what is found for $z_N \neq 0$. At this point, it is worth recalling that nonexponential decay has been found to emerge in different contexts; moreover, the potential use of this property in the implementation of the quantum Zeno effect has been widely analysed (see for instance [16] and references therein for a description of practical realizations of the effect).

(b) For $\nu_0 \neq 0, -1/2$, the numerical results show that, as predicted from the diffusive growth of the energy, $E^{(f)} = \frac{1}{4}\mathrm{Var}(\delta K)N$, noise gradually destroys the 'localization' found in the deterministic system (see figures 3(*c*) and (*d*). As the spreading progresses the nonexponential form of the distribution persists, as can be anticipated from the robustness of the functional form, $\left\langle\left|a_n^{(N)}\right|^2\right\rangle_f = \left\langle J_{n-n_0}^2(\eta_N)\right\rangle_f$.

We emphasize that, in the stochastic system, as in its deterministic counterpart, the momentum distribution has a discrete character since only components with $\rho = (n+\nu_0)\bar{k}$

are allowed. This 'momentum ladder' structure is rooted in the spatial periodicity of the potential, which is preserved by amplitude noise. It is also noticeable how, despite the time-increasing variance, the deterministic features are relatively robust, large times and/or large noise intensities being needed to observe their complete disappearance. Here, it is interesting to comment on some general considerations which allow us to anticipate the more destructive effect of other decohering mechanisms on the QR. Namely, the suppression of BM by spontaneous emission can be predicted: the transfer of a uniformly distributed random momentum at each spontaneous emission event makes ineffective the initial preparation needed for the appearance of BM (see [13] for a Monte Carlo simulation of spontaneous emission in the regime of dynamical localization). This is supported by the fact that a random choice of ν_0, i.e. a random initial condition, leads to a purely linear growth of the total mean energy. Furthermore, as BM is rooted in the nondispersive evolution between kicks allowed by the QR condition, its destruction by fluctuations in the kicking period, which prevent the exact fulfilment of this condition, can be anticipated.

4. Results for a Guassian initial distribution

The results presented up to now are strictly valid for a plane-wave initial state, $\psi_0(\phi) = \exp[i(n_0 + \nu_0)\phi]$. Let us now discuss their generalization to more realistic states. In the experiments of [3], where the quantum resonances were observed, the initial sample of atoms consisted of an ensemble distributed continuously in momentum. Following the theoretical analysis that explained the results of these experiments, we consider now that the initial state corresponds to a statistical mixture of plane-wave states with a Gaussian momentum distribution. Specifically, we assume that the probability of having a momentum ρ_0 at $t = 0$ is $F_0(\rho_0) = A\exp[-(\rho_0 - \rho_c)^2/2\sigma^2]$, where σ is the initial width of the distribution, ρ_c is the central momentum and A is a normalization constant. It is straightforwardly shown that, in the absence of noise, the distribution after N kicks is given by $F_N(\rho) = A\sum_m \exp[-(\bar{k}(m+\nu) - \rho_c)^2/2\sigma^2]J^2_{n-m}\left(\frac{\bar{K}}{\bar{k}}\frac{\sin N\beta}{\sin\beta}\right)$, where $\rho = (n+\nu)\bar{k}$ and $\beta \equiv \bar{k}\nu/2$. The effect of amplitude noise is readily evaluated through a direct application of the methodology previously presented. For generic noise, the momentum distribution reads

$$\langle F_N(\rho)\rangle_f = A\sum_m \exp[-(\bar{k}(m+\nu) - \rho_c)^2/2\sigma^2]\left\langle J^2_{n-m}\left(\eta_N^{(\nu)}\right)\right\rangle_f \tag{12}$$

where $\eta_N^{(\nu)}$ is obtained replacing β_0 by β, and z_N by $z_N^{(\nu)} \equiv \frac{\bar{K}}{\bar{k}}\frac{\sin N\beta}{\sin\beta}$, in equation (9). It is also direct to generalize our previous results for the energy. In particular, in the case of Gaussian white noise, we obtain

$$\langle\langle E(N)\rangle\rangle_f = \langle E(0)\rangle + A\int\left(\frac{\bar{K}}{2}\frac{\sin N\beta_0}{\sin\beta_0}\right)^2 \exp[-(\rho_0 - \rho_c)^2/2\sigma^2]\,d\rho_0 + \frac{1}{4}\mathrm{Var}(\delta K)N \tag{13}$$

where $\langle E(0)\rangle \equiv A\int d\rho_0 \frac{\rho_0^2}{2}\exp[-(\rho_0-\rho_c)^2/2\sigma^2]$ is the initial mean energy, the second term is the deterministic increment and the third term is the diffusive growth. Note that in the limit $\sigma \to 0$, the results for a plane-wave initial state are consistently recovered. From the above expression for the mean energy, it is evident that the conditions for the detection of ballistic growth are not more demanding in the noisy system than in its deterministic counterpart, where the effect has indeed been detected. In fact, in both cases, BM can be observed for an initial distribution conveniently centred at a momentum $\rho_c = (n_c + \nu_c)\bar{k}$ with $\nu_c = 0, -1/2$, and, importantly, with a sufficiently small initial width σ. Here, one should take into account that it is the central area of the distribution that contributes to the effect: BM is observed not only for a plane wave with $\nu_0 = 0, -1/2$, but also, at small times, for plane-wave initial states in the limit $\nu_0 \to 0, -1/2$. The same arguments can be applied to discuss the feasibility of detecting the features specific to coloured noise, which were shown to depend on the initial condition. Again, it is the magnitude of the statistical weight of the central area that determines the appearance of an effect associated with a specific value of ρ_c in the average behaviour of the distribution. Hence, we conclude that our findings for a plane-wave initial state, in particular, the persistence of quadratic growth in the stochastic system and effects due to noise colour, can also be observed when a statistical mixture with a sufficiently narrow momentum distribution is initially prepared; furthermore, they can be tested under working conditions similar to those corresponding to previous experiments on quantum resonances.

The relevance of our analysis to the applicability of the QR in schemes of coherent momentum enhancement must be remarked. The use of BM to coherently increase the momentum, apart from being limited by the nonuniform character of the acceleration in this regime, can be significantly restricted by the time-increasing destructive effect of amplitude noise on the coherent evolution. It is worth mentioning recent work where related issues have been tackled [22–26]. The results of [22] for a similar realization of the kicked rotor are particularly interesting; they show that a quite uniform acceleration can be achieved by orienting the standing wave in the gravity direction and working in *accelerator mode* regimes.

5. Concluding remarks

In summary, we have presented exact analytical results for the effect of amplitude noise on the δ-kicked rotor at QR. Noise has been shown to add a time growth to the deterministic energy and to induce a time increasing spreading in the momentum distribution. Although the ballistic peaks, characteristic of the noiseless distribution for particular initial states, broaden and eventually vanish, the associated BM persists; moreover, the 'survival probability' decays as N^{-1} at large times. On the other hand, the nonexponentially 'localized' distribution detected in the atomic realization of the model spreads gradually. Specific to white noise is a linear growth of energy; for coloured noise, a nonlinear initial transient and an asymptotic linear increase, the rate depending on the initial state and on the correlation time, are found. The generalization of these conclusions, which are strictly valid for a plane-wave initial state, to realistic states corresponding to the atom optics realization of the model has been discussed. An experimental test of our predictions seems feasible under standard working conditions. Finally, we remark that, apart from describing essential aspects of the experimental realization of the model, the study provides new elements to the analysis of decoherence: we have shown that nontrivial effects such as the persistence of intrinsically quantum properties in the stochastic dynamics or the selective efficiency of different decohering mechanisms in the destruction of quantum features can be tested on this ground.

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